Faculty Working Papers

A VINTAGE GROWTH MODEL FOR AN OPEN CITY

Jan K. Brueckner, Assistant Professor of Economics

#480

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

April 20, 1978

A VINTAGE GROWTH MODEL FOR AN OPEN CITY

Jan K. Brueckner, Assistant Professor of Economics

#480

Abstract:

This paper develops a vintage model of residential housing for an open city, where the utility level of residents is given by an exogenous function of time. Producers behave myopically in that they believe the future price per unit of housing services for their housing units will equal the current price. Demolition occurs when the expected present value of profits from continuing to operate existing structures equals the expected present value of profits from redevelopment. The model is analysed under the assumptions of Cobb-Douglas utility and production functions and constant rates of growth for income, commuting cost, the utility level, and the prices of non-land capital and agricultural land. Computer simulation of the model provides a concrete example of a city which grows according to the model.

To be presented at the conference on "Urban Transportation, Planning and the Dynamics of Land Use," The Transportation Center, Northwestern University, June 2-3, 1978.

A VINTAGE GROWTH MODEL FOR AN OPEN CITY

by

Jan K. Brueckner

The standard microeconomic theory of urban spatial structure implicitly postulates that the non-land capital in buildings is perfectly malleable. In effect, the theory says that a city is torn down and reconstructed every period, with production decisions embodying optimal responses to current prices. A number of recent papers have developed dynamic models of the housing stock which explicitly recognize the durability of structures.[1] Many of these studies emerged out of the recognition that static equilibrium models of residential housing, and the static urban spatial models based on them, are unrealistic as representations of existing cities. Unfortunately, these papers typically suffer from a variety of shortcomings. The analytical models have been exceedingly complex, and as a result, difficult to understand. Moreover, attempts to wed dynamic theory to a model of urban spatial structure have met with little success. The present paper contains a dynamic model which is relatively simple and has direct implications for urban spatial structure. The model relies heavily on a myopia assumption for housing producers, and its tractability is largely due to this assumption. While the spatial properties of the model are straightforward, computer simulation is used to provide a concrete example of an urban area which grows according to the model. In the first section of the paper, the theoretical model is developed, and in the second section, simulation results for the model are presented.

I.

In an open city, the utility level of residents is maintained at some exogenously given level by costless migration to and from the city. Letting the utility function for the (identical) residents of the city be $v(x,q)$, where x is the consumption level of a numeraire non-housing good and q is consumption of housing services, the open city assumption means that $v(x,q) = u(t)$, where u is a given function of time, t. This equation gives the combinations of x and q which yield the given exogenous utility level at any time t. Inverting the equation yields $x = x(q,u(t))$. A consumer living at distance k from the central business district (CBD) has the following budget constraint at time t:

$$x(q,u(t)) + pq + c(t,k) = y(t), \tag{1}$$

where $y(t)$ is income at t, $c(t,k)$ is commuting cost to the CBD from k at t, and p is the unit price of housing services. Solving for p yields

$$p = \frac{y(t) - c(t,k) - x(q,u(t))}{q} \tag{2}$$

Define $p(t,k)$ to be the maximum price per unit of housing services consistent with the budget constraint and the exogenous utility level. That is,

$$p(t,k) \equiv \max_{\{q\}} \frac{y(t) - c(t,k) - x(q,u(t))}{q} \tag{3}$$

The maximizing q satisfies

$$\frac{-\partial x(q,u(t))}{\partial q} = \frac{y(t) - c(t,k) - x(q,u(t))}{q}, \tag{4}$$

and we denote it $q(t,k)$. It is easy to see that the budget line based on

the price $p(t,k)$ is just tangent to the indifference curve with utility level $u(t)$ at $q = q(t,k)$.

Consider a housing producer who is deciding how to divide a building into apartment units. He has a certain total number of square feet to rent, but the price per square foot (per unit of housing services) depends on the size of the units into which he divides the building. He maximizes the price per square foot (per unit of housing services), and hence maximizes his current revenue from the building, by dividing the building into units of size $q(t,k)$. Let the constant-returns-to-scale production function for housing services be $H(N,\ell)$, where N is non-land capital and ℓ is land. The above argument says that for any inputs of N and ℓ, current revenue is maximized by dividing the output $H(N,\ell)$ into units of size $q(t,k)$, with maximized current revenue given by $p(t,k)H(N,\ell)$.

Housing units deteriorate over time, and, as is explained fully below, the price per unit of services for a housing unit will change as income, transportation cost, and the utility level change over time and the services provided by the unit diminish. Our myopia assumption for housing producers states that producers believe that the price per unit of services for their housing units will equal its current value in all future periods. The myopia assumption implies that producers will always construct new housing units to maximize the current price per unit of services for those units. Considerable uncertainty surrounds the future time path of the price per unit of services; predicting it perfectly would require precise knowledge of the time paths of the exogenous variables in the urban economy. In the face of this uncertainty, producers act myopically and base their production decisions on a price assumption which will not be validated by future experience.

Letting α be the exponential rate of decline of housing services provided by a unit, the housing services provided at time t by a unit constructed optimally at t_0 are $q(t_0,k)e^{-\alpha(t-t_0)}$. Under myopia, the revenue expected at time t from $H(N,\ell)$ worth of housing services constructed at t_0 and divided optimally equals $p(t_0,k)H(N,\ell)e^{-\alpha(t-t_0)}$. The expected present value of revenue between $t_0 \div T$ from an optimally-divided output $H(N,\ell)$, new at t_0, is

$$\int_{t_0}^{t_0+T} p(t_0k)H(N,\ell)e^{-(\alpha+r)(t-t_0)}dt = \frac{p(t_0,k)H(N,\ell)}{\alpha+r}(1-e^{-(\alpha+r)T}), \quad (5)$$

where r is the discount rate.

We assume that housing producers building new units at t_0 purchase N at a unit price $n(t_0)$ and purchase land at a price per acre of $R(t_0,k)$. Other investigators have assumed that housing producers rent the land they use, but for our purposes it will be convenient to assume they purchase it. Producers borrow to finance their input costs, and arrange to repay the loans in equal payments over the expected life of their structure. The present value of the repayment streams at the time of construction must equal the cost of the inputs, $n(t_0)N + R(t_0,k)\ell$. Since this present value is independent of the length of the life of the structure while the expected present value of revenue increases with the length of the horizon T from (5), the producer sets $T = \infty$ and arranges a repayment stream of infinite length, with payments of $rn(t_0)N + rR(t_0,k)\ell$ per period. Using (5) with $T = \infty$, the expected present value of profits under myopia from new construction at t_0 equals

$$\frac{p(t_0,k)H(N,\ell)}{\alpha+r} - n(t_0)N - R(t_0,k)\ell. \quad (6)$$

Producers choose N and ℓ to maximize (6), and Euler's theorem guarantees that the maximized expected present value of profits is zero. The first-order conditions for maximizing (6) are $pH_N/(\alpha+r) = n$ and $pH_\ell/\alpha+r = R$, but with constant returns, only the ratio N/ℓ can be solved for, and either of these conditions gives the solution. Using the fact that the first partial derivatives of a function homogeneous of degree one are homogeneous of degree zero, we may write the first condition as

$$\frac{p(t_0,k)}{\alpha+r} H_N(N/\ell,1) = n(t_0) \ . \tag{7}$$

Denoting the solution to (7) N_0/ℓ_0 and using the zero profit condition, land rent is given by

$$R(t_0,k) = \frac{p(t_0,k)}{\alpha+r} H(N_0/\ell_0,1) - n(t_0)N_0/\ell_0 \ . \tag{8}$$

The actual price per unit of services at time t for a housing unit optimally constructed at time t_0 is denoted $p(t,k;t_0)$, and from (2) this can be written

$$p(t,k;t_0) = \frac{y(t) - c(t,k) - x(q(t_0,k)e^{-\alpha(t-t_0)}, u(t))}{q(t_0,k)e^{-\alpha(t-t_0)}} \ , \qquad t \geq t_0 \tag{9}$$

Note that $p(t_0,k;t_0) \equiv p(t_0,k)$. Equation (9) gives the price per unit of services for a consumer inhabiting a unit providing service level $q(t_0,k)e^{-\alpha(t-t_0)}$ which results in satisfaction of the budget constraint and generates the correct utility level. Recall that our myopia assumption states that when the producer estimates future revenue from his units, he assumes that the price per unit of services will continue at its current level. At time t_1, the owner of units built optimally at time t_0 receives

$p(t_1,k;t_0)$ per unit of housing services for those units, and with myopia he believes that he will continue to receive this price per unit of services in future periods. His expected present value of profits at t_1 under this assumption is

$$\int_{t_1}^{\infty} (p(t_1,k;t_0)H(N_0,\ell_0)e^{-\alpha(t-t_0)} - rn(t_0)N_0 - rR(t_0,k)\ell_0)e^{-r(t-t_1)}dt$$

$$= \frac{p(t_1,k;t_0)H(N_0,\ell_0)}{\alpha+r} e^{-\alpha(t_1-t_0)} - n(t_0)N_0 - rR(t_0,k)\ell_0. \qquad (10)$$

The first term is the expected present value of revenue from a structure which provides housing service level $H(N_0,\ell_0)e^{-\alpha(t_1-t_0)}$ at the start of the period and for which the price per unit of services is expected to equal $p(t_1,k;t_0)$ indefinitely. Because of the infinite horizon, the present value of the remaining repayment stream equals the initial cost of the inputs.

Producers will wish to redevelop their property when the expected present value of profits from redevelopment on the original land exceeds the expected present value of profits from continuing to operate the original structures. At t_1, the former is

$$\max_{\{N\}} \int_{t_1}^{\infty} (p(t_1,k)H(N,\ell_0)e^{-\alpha(t-t_1)} - rn(t_1)N - rR(t_0,k)\ell_0)e^{-r(t-t_1)}dt$$

$$- n(t_0)N_0 - D(t_1)\ell_0 \,. \qquad (11)$$

$D(t_1)$ is demolition cost per acre at t_1, and $n(t_0)N_0$ appears in (11) because the producer must still pay $rn(t_0)N_0$ per period on the non-land capital used in the original buildings, which is destroyed when the buildings are demolished. Now (11) is greater than or equal to (10) when

$$\max_{\{N\}} \left(\frac{p(t_1,k)H(N/\ell_0,1)}{\alpha+r} - n(t_1)\ N/\ell_0 \right) - D(t_1) \geq$$

$$\frac{p(t_1,k;t_0)H(N_0/\ell_0,1)}{\alpha+r}\, e^{-\alpha(t_1-t_0)} \tag{12}$$

But from above, the first term on the LHS of (12) is $R(t_1,k)$, and this means that the producer will wish to redevelop at t_1 if

$$R(t_1,k) - D(t_1) \geq \frac{p(t_1,k;t_0)H(N_0/\ell_0,1)}{\alpha+r}\, e^{-\alpha(t_1-t_0)}. \tag{13}$$

This condition is intuitively sensible because it states that redevelopment will occur if the expected present value of revenue per acre of land from the original structures falls short of land's resale price per acre less demolition cost per acre. Given appropriate conditions on the functions in (13), there will exist a unique t^*, which depends in general on t_0 and k, when redevelopment occurs; for $t_0 < t_1 < t^*$, the LHS of (13) will fall short of the RHS, and for $t_1 > t^*$ the LHS will exceed the RHS.

We now impose specific forms for the utility and housing production functions and the functions $y(t)$, $c(t,k)$, $u(t)$, and $n(t)$ in order to make analysis of the model tractable. First, it is assumed that the housing production function H is Cobb-Douglas: $H(N,\ell) \equiv N^{\beta}\ell^{1-\beta}$. Then (7) gives

$$\frac{\beta p(t_0,k)}{\alpha+r}\left(\frac{N}{\ell}\right)^{\beta-1} = n(t_0) \tag{14}$$

and substituting into (8) and rearranging yields

$$R(t_0,k) = (1-\beta)\left[\frac{p(t_0,k)}{\alpha+r}\left(\frac{\beta}{n(t_0)}\right)^{\beta}\right]^{\frac{1}{1-\beta}} \tag{15}$$

Using (14) to find $H(N_0/\ell_0,1)$, (13) may be rewritten

$$(1-\beta)\left[\frac{p(t_1,k)}{\alpha+r}\left(\frac{\beta}{n(t_1)}\right)^{\beta}\right]^{\frac{1}{1-\beta}} - D(t_1) \geq$$

$$\frac{p(t_1,k;t_0)}{\alpha+r}\left[\frac{\beta\, p(t_0,k)}{(\alpha+r)n(t_0)}\right]^{\frac{\beta}{1-\beta}} e^{-\alpha(t_1-t_0)} . \tag{16}$$

The demolition date t^* will be the value of t_1 for which (16) holds as an equality. Rearranging (16) and assuming demolition costs are zero, we have

$$\frac{(1-\beta)p(t^*,k)}{p(t^*,k;t_0)}\left[\frac{p(t^*,k)}{p(t_0,k)}\right]^{\frac{\beta}{1-\beta}} = \left[\frac{n(t_0)}{n(t^*)}\right]^{\frac{-\beta}{1-\beta}} e^{-\alpha(t^*-t_0)} . \tag{17}$$

Assuming a Cobb-Douglas utility function, $v(x,q) \equiv x^{\theta}q^{1-\theta}$, gives $x = q^{(\theta-1)/\theta}\, u(t)^{1/\theta}$ and

$$p(t,k) = \max_{\{q\}} \frac{y(t) - c(k,t) - q^{\frac{\theta-1}{\theta}} u(t)^{\frac{1}{\theta}}}{q} \tag{18}$$

The maximizing q, $q(t,k)$, satisfies

$$\begin{aligned} q(t,k)^{\frac{\theta-1}{\theta}} u(t)^{\frac{1}{\theta}} &= \theta(y(t) - c(t,k)) \\ q(t,k) &= (\theta(y(t) - c(t,k))\, u(t)^{-1/\theta})^{\frac{\theta}{\theta-1}} \end{aligned} \tag{19}$$

so that

$$p(t,k) = (1-\theta)\theta^{\frac{\theta}{1-\theta}}\, u(t)^{-\frac{1}{1-\theta}}\, (y(t) - c(t,k))^{\frac{1}{1-\theta}} \tag{20}$$

We assume that $c(t,k) \equiv c(t)k$, that both this function and $y(t)$ grow exponentially at the same rate y, and that the utility level grows at the exponen-

tial rate u, yielding $y(t) - c(t,k) = (y_0 - c_0 k)e^{yt}$ and $u(t) = u_0 e^{ut}$. These assumptions yield

$$p(t,k) = (1-\theta)\theta^{\frac{\theta}{1-\theta}} \left(\frac{y_0 - c_0 k}{u_0}\right)^{\frac{1}{1-\theta}} e^{\frac{(y-u)t}{1-\theta}} \equiv A(k)e^{\frac{(y-u)t}{1-\theta}} . \tag{21}$$

Since $A'(k) < 0$, $\partial p(t,k)/\partial k < 0$, and $\partial p(t,k)/\partial t \gtrless 0$ as $y \gtrless u$. The price per unit of housing services in terms of the numeraire x in newly-constructed units increases (decreases) over time when the percentage rate of increase of income is greater than (less than) the percentage rate of increase of the utility level. Defining $q(t,k;t_0)$ to be $q(t_0,k)e^{-\alpha(t-t_0)}$, we have

$$q(t,k;t_0) = (\theta(y(t_0) - c(t_0,k))u(t_0)^{-1/\theta})^{\frac{\theta}{\theta-1}} e^{-\alpha(t-t_0)} . \tag{22}$$

From (9), we have

$$p(t,k;t_0) = \frac{y(t) - c(t,k) - q(t,k;t_0)^{\frac{\theta-1}{\theta}} u(t)^{\frac{1}{\theta}}}{q(t,k;t_0)} \tag{23}$$

which, using (22) and the maintained assumptions on the functions y, c, and u, reduces to

$$p(t,k;t_0) = \frac{A(k)}{1-\theta} \exp[(y+\alpha)t - (\frac{u}{1-\theta} + \alpha - \frac{\theta y}{1-\theta})\ t_0] \cdot$$

$$(1 - \theta \exp[(\frac{u}{\theta} + \frac{1-\theta}{\theta}\alpha - y)(t-t_0)]) \tag{24}$$

Since $A'(k) < 0$, $\partial p(t,k;t_0)/\partial k < 0$. Now $\partial p(t,k;t_0)/\partial t$ has the same sign as

$$(y+\alpha) - (u+\alpha) \exp[(\frac{u}{\theta} + \frac{1-\theta}{\theta}\alpha - y)(t-t_0)] . \tag{25}$$

If $y \leq u$, then the argument of exp is positive, exp(-) is greater than one, and in view of $y + \alpha \leq u + \alpha$, (25) is negative. This means that when $y \leq u$, the price per unit of services for optimally-constructed housing

units declines continuously as they age. If $u < y < (u+(1-\theta)\alpha)/\theta$, then $\exp(-)$ is increasing with t, and although (25) is positive for t near t_0, as t becomes large $\partial p(t,k;t_0)/\partial t$ becomes negative. The price per unit of services for an optimally-constructed housing unit increases at first but eventually must decrease with age when $u < y < (u+(1-\theta)\alpha)/\theta$. Finally, for $y \geq (u+(1-\theta)\alpha)/\theta$, $p(t,k;t_0)$ is monotonically increasing in t. Similarly, $\partial p(t,k;t_0)/\partial t_0$ has the same sign as

$$(\frac{u}{\theta} + \frac{1-\theta}{\theta}\alpha - y)\ (\frac{\theta}{1-\theta}\exp[(\frac{u}{\theta} + \frac{1-\theta}{\theta}\alpha - y)\ (t-t_0)] - 1)\ . \qquad (26)$$

Regardless of the sign of $((u+(1-\theta)\alpha)/\theta) - y$, (26) and $\partial p(t,k;t_0)/\partial t_0$ are positive when $t-t_0$ is sufficiently large. That is, at a given time, the price per unit of services for old housing units will decrease as the age of units increases, provided we start back far enough in the age distribution.

From (21) and (15), the land price function is

$$R(t,k) = (1-\beta)(\frac{\beta}{n_0})^{\frac{\beta}{1-\beta}}\ (\frac{A(k)}{\alpha+r})^{\frac{1}{1-\beta}}\ \exp[\frac{y-u-n\beta(1-\theta)}{(1-\beta)(1-\theta)}\ t]\ , \qquad (27)$$

which increases in t only if $y-u > n\beta(1-\theta)$. Since $A'(k) < 0$, we have $\partial R(t,k)/\partial k < 0$.

We now analyse (17), which determines the demolition date of buildings. First, we assume $n(t) = n_0 e^{nt}$. Postive n means the price of N in terms of the numeraire increases over time, while negative n implies a decreasing price. Under this assumption, the RHS of (17) becomes $\exp[-(\alpha-n\beta/(1-\beta))(t^*-t_0)]$, and the LHS becomes, using (21) and (24),

$$(1-\theta)(1-\beta)\ \exp[\frac{\beta(y-u)}{(1-\beta)(1-\theta)}\ (t^*-t_0)]\cdot\frac{\exp[B(t^*-t_0)]}{1-\theta\ \exp[-\frac{1-\theta}{\theta}\ B(t^*-t_0)]}, \qquad (28)$$

where $B = (\theta y-u)/(1-\theta) - \alpha$. Letting z represent t^*-t_0, which equals the age of buildings at their demolition date, (17) can be rewritten as

$$\frac{(1-\theta)(1-\beta)\ \exp[(C+B)z]}{1-\theta\ \exp[-\frac{1-\theta}{\theta}Bz]} = 1, \qquad (29)$$

where $C = (\beta(y-u-n(1-\theta))/(1-\beta)(1-\theta)) + \alpha$. An important result is that since (29) does not involve t_0, a building's age at demolition does not depend on its construction date.

The LHS of (29) evaluated at $z = 0$ equals $1-\beta$, which is less than the RHS. If the LHS decreases in z, then (29) will never hold for positive z, and buildings will deteriorate indefinitely. If the LHS is increasing in z, then (29) will hold for some positive z, and buildings will have a finite life. Detailed analysis of (29) turns out to be fairly involved because of the ambiguity of the signs of C and B. If $B < 0$, then the denominator of the LHS of (29) is decreasing in z and assumes the value zero at $z = \hat{z} \equiv \theta\log(1/\theta)/(\theta-1)B$. Since the numerator is positive and bounded for $0 \leq z \leq \hat{z}$, the LHS of (29) approaches $+\infty$ as z approaches $\hat{z}$ from below. Therefore, at least one z between 0 and $\hat{z}$ exists which satisfies (29) when $B < 0$. Whether the solution is unique depends on the sign of C. For $z \neq \hat{z}$, the derivative of the LHS of (29) has the same sign as

$$C(1-\theta\ \exp[-\frac{1-\theta}{\theta}Bz]) + B(1-\exp[-\frac{1-\theta}{\theta}Bz]). \qquad (30)$$

From above, the coefficient of C in (30) is positive for $0 \leq z \leq \hat{z}$. If $B < 0$ then the last expression in (30) is positive for $z > 0$ because

$1 - [\exp \frac{-(1-\theta)}{\theta} Bz] < 0$. Therefore if $C > 0$, (30) is positive, implying that the LHS of (29) is monotonically increasing in z between 0 and $\hat{z}$ and that a unique solution z^* exists. If $C < 0$, (30) cannot be signed unambiguously, and the existence of multiple solutions to (29) cannot be ruled out. It should be noted that (29) can never be satisfied for $z > \hat{z}$ because the LHS is negative in this range.

When $B > 0$, the LHS of (29) is never discontinuous since the denominator never equals zero. If $C > 0$ there exists a unique positive solution to (29), because both terms in (30) are positive for $z > 0$ and hence (29) is monotonically increasing when z is positive. When $C > 0$, it is useful to rewrite (30) as

$$(C+B)\left(1 - \frac{\theta C+B}{C+B} \exp\left[-\frac{1-\theta}{\theta}Bz\right]\right). \tag{31}$$

When $C < 0$, the sign of (31) depends on the sign of C+B. If $C+B > 0$, then $(\theta C+B)/(C+B) > 1$ and (31) is ambiguous in sign. However, since exp(-) approaches zero as z increases, (31) assumes the positive sign of C+B for z large enough. Thus we can be sure that a solution to (29) exists although it may not be unique. If $B > 0$, $C < 0$ and $C+B < 0$, then $(\theta C+B)/(C+B) < 1$ and (31) has the negative sign of C+B for $z > 0$. Thus the LHS of (29) decreases monotonically when $z > 0$; no retirement age exists and buildings are never demolished. Finally, we note that when $C = 0$ and $B \neq 0$, a unique solution to (29) exists ((30) is positive for $C = 0$ when $B \neq 0$), and when $B = 0$, a unique solution exists if $C > 0$ while no retirement age exists if $C \leq 0$. Summarizing these results, we conclude that a solution to (29) exists, although it may not be unique,

unless B > 0, C < 0, and C+B < 0, or B = 0 and C $\leq$ 0. Note that a simple sufficient condition for the existence of a unique solution is C > 0, a condition which was satisfied in the simulation reported below.

Consideration of (10) and (11) shows that it is conceivable that the expected present value of profits from buildings could turn negative before they are demolished. Indeed, it is even conceivable that the producer suffers current losses but continues to operate the buildings. To investigate these issues, we perform the following analysis. The expected present value of profits is positive at t* when

$$\int_{t^*}^{\infty} (p(t^*,k;t_0)H(N_0,\ell_0)e^{-\alpha(t-t_0)} - rn(t_0)N_0 - rR(t_0,k)\ell_0)e^{-r(t-t^*)}dt$$

$$> 0 = \int_{t_0}^{\infty} (p(t_0,k)H(N_0,\ell_0)e^{-\alpha(t-t_0)} - rn(t_0)N_0 - rR(t_0,k)\ell_0)e^{-r(t-t_0)}dt ,$$

where the equality follows because the expected present value of profits at the construction date is zero. Now (32) reduces to

$$p(t^*,k;t_0)e^{-\alpha(t^*-t_0)} > p(t_0,k) . \tag{33}$$

Similarly, positivity of current profit at t* means

$$p(t^*,k;t_0)H(N_0,\ell_0)e^{-\alpha(t^*-t_0)} - rn(t_0)N_0 - rR(t_0,k)\ell_0 > 0$$

$$\frac{p(t^*,k;t_0)H(N_0/\ell_0,1)}{r} e^{-\alpha(t^*-t_0)} - n(t_0)N_0/\ell_0 > R(t_0,k) \tag{34}$$

$$= \frac{p(t_0,k)H(N_0/\ell_0,1)}{\alpha+r} - n(t_0)N_0/\ell_0,$$

using (8). But this reduces to

$$\frac{(\alpha+r)}{r}\, p(t^*,k;t_0)e^{-\alpha(t^*-t_0)} > p(t_0,k) \tag{35}$$

Positive expected present value of profits clearly implies positive current profits, but the converse is not true. Using (17) to substitute for $p(t^*,k;t_0)$, (35) reduces to

$$\exp[\frac{y-u-n\beta(1-\theta)}{(1-\beta)(1-\theta)}\,(t^*-t_0)] > \frac{r}{\alpha+r}\,\frac{1}{1-\beta}\,, \tag{36}$$

and (33) reduces to (36) with $r/(\alpha+r)$ replaced by unity. If (36) holds at t^*, then both current profits and the expected present value of profits from a building are positive when it is demolished. While there is no brief sufficient condition for the satisfaction of either (36) or (36) with $r/(\alpha+r)$ replaced by unity, it is easy to see that the latter inequality will fail to hold at t^* if $y-u < n\beta(1-\theta)$. Noting (27), this means that when land prices are falling over time, the expected value of profits from a building is negative at its demoliton date.

Taking a different approach, we note that since the LHS of (33) equals the RHS for $t^*=t_0$, (33) (and also 35) will hold for all $t^* > t_0$ if the LHS of (33) is increasing in t^*. The derivative of this quantity has the same sign as

$$y - (u+(1-\theta)\alpha)\, \exp[(\frac{u}{\theta} + \frac{1-\theta}{\theta}\,\alpha-y)\, (t^*-t_0)]\,, \tag{37}$$

and a sufficient condition for (37) to be positive is $y \geq (u+(1-\theta)\alpha)/\theta$. When this inequality holds, both the expected present value of profits and current profits are always positive for aging structures, being positive, in particular, at the demolition date.

The spatial implications of our model are straightforward. Suppose at t=0, an open city is constructed. At t=0, the city is indistinguishable from cities described by the static equilibrium urban model; the urban periphery $\overline{k}_0$ is given by $R(0,\overline{k}_0) = R^A(0)$, where R^A is the agricultural land price function, and the price of housing services, population density, and building heights decline as distance to the CBD increases. Suppose $y-u > n\beta(1-\theta)$, so that the urban land price at any given distance increases over time. Suppose also that $R^A(t) \equiv R_0^A\, e^{gt}$ and that g is less than the percentage rate of increase of R(t,k) from (27). This means that for $t > 0$, $R(t,\overline{k}_0) > R^A(t)$, and this implies that housing producers can outbid agricultural users for land beyond $\overline{k}_0$ as t increases away from zero. Thus, the distance to the boundary of the city increases, with new structures being built at the periphery. Formally, writing R(t,k) from (27) as $F(k)e^{Gt}$, with $F'(k) < 0$, $\partial\overline{k}/\partial t$ for a growing city is determined by totally differentiating $F(\overline{k})e^{Gt} = R_0^A e^{gt}$. This yields

$$\frac{\partial\overline{k}}{\partial t} = \frac{gF(\overline{k})(1-G/g)}{F'(\overline{k})}\,, \tag{37}$$

which is positive for $G > g$. Changes in $\overline{k}$ for a city which shrinks in area over time are more difficult to analyse, as is shown below.

Eventually, the part of the city built at time zero is demolished and rebuilt optimally, and the narrow rings of structures that were built as the city expanded meet the same fate. At certain times in the city's development, structures close to the CBD will be younger than structures near the periphery. The nature of this growth process suggests that at certain points in the city's development, its spatial structure may be quite different from that predicted by the static equilibrium model. It is

conceivable that the unit price of housing, population density, and building heights may increase over certain ranges of distance as we move away from the CBD.

Suppose, contrary to our previous assumption, that urban land rent decreases over time. Then as t increases from zero, $R(t,\bar{k}_0) < R^A(t)$ holds, and no expansion of the urban area takes place. At first we might be tempted to think that the following events take place. Let t_0^* be the time from (17) (which is independent of k) at which redevelopment is supposed to take place. At t_0^*, there will be some $\bar{k}^* < \bar{k}_0$ satisfying $R(t_0^*,\bar{k}^*) = R(t_0^*)$. We might believe that the entire original city is torn down at t_0^*, with the land between $\bar{k}^*$ and $\bar{k}_0$ sold to agricultural users and the area between $k=0$ and $\bar{k}^*$ redeveloped. This claim is not completely correct because we can show that housing producers between $\bar{k}^*$ and $\bar{k}_0$ actually demolish their buildings and sell the land to agricultural users at times before t_0^*. To see this, consider (13) with $D(t_1) = 0$. When producers consider the option of selling their land to agricultural users, the appropriate LHS for (13) is $\max\{R(t_1,k),R^A(t_1)\}$. If the larger of urban and agricultural land prices exceeds the RHS of (13), then the buildings are demolished and the land is either redeveloped or sold to an agricultural user, depending on which argument of max is greater. We know that for $t_1 = t_0^*$, $R(t_0^*,k)$ equals the RHS of (13) evaluated at $t_1 = t_0^*$ for $0 \leq k \leq \bar{k}_0$. But we know that $R(t_0^*,k) < R^A(t_0^*)$ for $\bar{k}_0 < k < \bar{k}^*$. Hence $R^A(t_0^*)$ is greater than the RHS of (13) for k in this range. This means that for producers between $\bar{k}^*$ and $\bar{k}_0$, inequality (13) with $\max\{R(t_0^*,k),R^A(t_0^*)\}$ on the LHS holds strictly,

implying that these producers should have demolished their buildings and sold the land sometime before t_0^*.

It is interesting to consider what might be called balanced growth for an open city, where the rates of growth of income and utility are equal, and where no exogenous relative prices change. The latter condition means that $n = g = 0$, or that the prices of non-land capital and agricultural land in terms of the numeraire do not change over time. These assumptions mean that $p(t,k)$ and $R(t,k)$ are constant functions of time (see (21) and (27)), and that $\partial\bar{k}/\partial t = 0$ (see (37)), which says that the area of the city does not change over time. Since no prices change with time, the city is periodically torn down and rebuilt exactly as it was originally constructed, and the urban population stays constant at its original level.

II.

In this section of the paper, simulation of the model developed in Part I is discussed. The model was simulated twice with slightly different parameter values. In both simulations, β, the exponent of N in the housing production function, equals .75; θ, the exponent of x in the utility function, equals .6; α, the rate of deterioration of structures, equals .01; y, the rate of growth of income, equals .04; n and g, the rates of growth of the prices of non-land capital and agricultural land respectively, equal zero; and u_0, the utility level at time zero, equals one. In the first simulation, u, the rate of growth of the utility level, equals .03, while in the second it equals .01. Since $y-u > 0$ in both cases, $p(t,k)$ and $R(t,k)$ are increasing in t (see (21) and (27)), and the latter fact implies that the city grows in area over time.

It was assumed that housing production occurs only at the beginning of each period, or at integer values of t. This means that the life of buildings is integer-valued, being equal to the first integer z for which the LHS of (29) is greater than or equal to unity (our parameter value assumptions give $C > 0$, which means, from above, that the LHS of (29) is monotonically increasing in z). In the first simulation, buildings are demolished after 15 years of life, while in the second, they are demolished after 8 years of life. For example, in the second case, this means that buildings constructed at the beginning of period zero are replaced at the beginning of period eight, and so on. While other parameter values lead to more realistic 45-50 year lifespans for structures, our short lifespans meant that the time span of the simulations could be relatively short.

Since units of distance are arbitrary, they were chosen so that the cost of traveling a unit of distance at time zero, c_0, was equal to some specified value. We refer to these units of distance as "blocks." It was assumed that the city grows outward at its periphery by the addition of concentric rings one block wide. Since construction occurs only at integer values of time, new rings are added only at integer times. In order for a ring to be bid away from agricultural users, the value of the urban land price function at the ring's outer radius had to exceed the value of agricultural land rent. All locations within a given ring were taken to be equidistant from the CBD, the distance being equal to the outer radius of the ring. This implies that quantities which vary over distance will differ between but not within rings.

To compute the spatial growth of the city, (21), (27), and the parameter assumptions were used to write the condition $R^A(t) = R(t,\bar{k})$ as

$$R_0^A = m((y_0 - c_0\bar{k})e^{(y-u)t})^{1/(1-\theta)(1-\beta)} , \tag{38}$$

where m is a constant which depends on n_0. Solving (38) for $\bar{k}$ gives

$$\bar{k} = \frac{1}{c_0}(y_0 - (\frac{m}{R_0^A})^{(1-\theta)(1-\beta)} e^{-(y-u)t}) \tag{39}$$

In light of the above discussion, the radius of the city at a particular time equals the largest integer less than or equal to the RHS of (39) evaluated at the given time.

One requirement in the simulations was that the city start out relatively small at time zero and grow to an appreciably larger size by a time roughly equal to a small multiple of the life of buildings. In addition to guaranteeing adequate spatial growth, this requirement allowed observation of several generations of structures in the simulations. Satisfying this requirement meant choosing appropriate values for y_0, c_0, and m/R_0^A. Since the latter constant depends on n_0 through m, its value may be set at any level by appropriate choices of n_0 and R_0^A. For the first simulation, we chose $y_0 = 110$, $c_0 = 1$, and $(m/R_0^A)^{(1-\theta)(1-\beta)} = 100$, which yielded a city with a radius of 10 blocks at time zero and 49 blocks at time 50. For the second simulation, choosing $y_0 = 50$, $c_0 = .5$, and $(m/R_0^A)^{(1-\theta)(1-\beta)} = 47.5$ yielded a city with radius 5 at time zero and radius 48 at time 30.

The quantities computed in the simulation were $p(t,k;t_0)$ from (24), population density, urban population, and the non-land capital to land

ratio in structures, N/ℓ, which represents the height of buildings. The above analysis showed that the unit price of housing services changes continuously in structures as they age. However, it is clear that while the services provided by a unit diminish over time, the physical size of the unit does not change over its lifetime. Thus, population density as well as N/ℓ are constant over the lifetime of structures. Using (14) and (21), $N/\ell(t,k;t_0)$, which is defined to be the N/ℓ ratio at time t in structures built at time t_0 at distance k from the CBD, is given by

$$N/\ell(t,k;t_0) = \left(\frac{(\alpha+r)n_0}{\beta A(k)}\right)^{-\frac{1}{1-\beta}} \exp\left[\frac{y-u}{(1-\theta)(1-\beta)} t_0\right] , \tag{40}$$

which is independent of t but increasing in t_0 under our assumptions. $D(t,k;t_0)$, population density at time t in structures built at time t_0 at distance k from the CBD, equals $(N/\ell(t,k;t_0))^{\beta}/q(t_0,k)$, or output of housing services per acre divided by housing services per new unit. Using (40) and (19), we have

$$D(t,k;t_0) = \left(\frac{\beta(1-\theta)}{n_0(\alpha+r)}\right)^{\frac{\beta}{1-\beta}} \theta^{\frac{\beta}{(1-\beta)(1-\theta)}} (y_0 - c_0 k)^{\frac{\beta+\theta(1-\beta)}{(1-\theta)(1-\beta)}} \cdot$$

$$\exp\left[\frac{(\beta+\theta(1-\beta))y-u}{(1-\theta)(1-\beta)} t_0\right] , \tag{41}$$

which is also independent of t and increasing in t_0 under our assumptions. Since population density is constant within each ring, ring population was computed by multiplying the area of the ring by the population density at the ring's outer boundary. Total urban population at a given time was

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

computed by summing the population of all occupied rings. Each computed variable was normalized by multiplying by 100 and dividing by the variable's value at t=0 and k=0 (or simply at t=0 in the case of population). While the normalization allows easy comparison of variable values over time, the principal reason for it was to eliminate the need for an arbitrary choice of n_0, which sets the levels of N/ℓ and population density.

Tables Ia-f give the computed values of the variables for the first simulation at times 0, 13, 22, 33, 40, and 50, while Tables IIa-e give the values for the second simulation at times 0, 7, 14, 25, and 30. While the variables in the two simulations were computed for all integer times between 0 and 50 and 0 and 30 respectively, the partial results reported in the Tables are sufficient to show the nature of the urban growth process.

While the results are mostly self-explanatory, some observations may be useful. At time zero, both cities have the same properties as cities generated by the static equilibrium model. Tables Ia and IIa show that building heights, population density, and the unit price of housing services decline monotonically with distance to the CBD. In Table Ib, which shows the first city at time 13, all the original buildings are 13 years old, and the ages of the remaining buildings decline monotonically as we move away from the CBD. The reason for the skip from 13 to 11 at block 11 in the age column is that the city did not grow at t=1, but added its first ring at t=2. Similar skips in the ages of adjacent buildings which occur in the Tables are also due to the existence of periods in which the cities did not grow. Further examination of Table Ib shows a jump in the height of buildings, population density, and the price of housing at block 11,

with smooth declines thereafter. In Table Ic, the original area of the city has its second generation of structures. Some ranges of increasing building heights and population density are evident, and large jumps in all the variables are evident at block 17, where new buildings are adjacent to 14-year-old structures. In Table Id, the original city has third-generation structures, and ranges of increasing building heights and density, in addition to large jumps in the variables in blocks where the age of structures changes dramatically, are evident. In Figure 1, the base-10 logarithms of N/ℓ and density from Table Id are graphed. Tables Ie and If show that the general pattern of growth in the early Tables persists as time progresses. Note that the urban population at time 50 is almost 53 times as great as its original level.

Tables IIa-e show that the general features of growth in the second simulation are similar to those in the first. Since u is smaller in the second simulation, building heights, density, and population grow faster than in the first. Ranges of increasing building heights and density are more common in the second than in the first simulation, as the graphs of the logs of N/ℓ and density from Table IId, shown in Figure 2, make clear. Note that since in the first simulation the value of c_0, the cost of traveling a unit of distance, is twice its value in the second, a block in Figure 2 is only half as long as a block in Figure 1. Note also that the urban population at the end of the second simulation is more than 718 times as large as its original level.

Examining Figures 1 and 2, we can make some generalizations about the spatial properties of our growing cities. As long as the age of buildings declines monotonically, the N/ℓ and density curves are more

or less gradually declining (Figure 1) or roughly flat (Figure 2). Large decreases in the values of the variables are associated with abrupt changes in the age of the buildings. In contrast to the static urban model, it is the changing age of buildings as we move away from the CBD rather than distance to the CBD per se which accounts for the large decreases in population density and building heights.

Our results are satisfying in that they mimic the spatial properties of many existing cities. In cities where "Manhattanization" is occuring, we typically observe a core of new skyscrapers which are very much taller than the surrounding older structures. In addition, in some cities we observe ranges of increasing building heights as distance from the CBD increases. Both these phenomena can be generated by our model, as the Figures show.

The model in this paper has provided for the first time a satisfactory integration of a vintage model of urban housing and a spatial model of urban growth. The open city assumption which underlies the model is one reason for its tractability. Requiring that the urban utility level equal some exogenous function of time allowed application of analytical techniques from static urban economic models. A much more difficult task for future research is the construction of a vintage growth model for a closed city, where utility is endogenous and population is given by an exogenous function of time. Preliminary investigation suggests that the closed city problem is more difficult by several orders of magnitude than the one solved in this paper.

Table Ia: Time Equals 0
Population = 100

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	100.00	100.00	100.00	0
1	91.27	92.11	97.74	0
2	83.24	84.78	95.52	0
3	75.84	77.97	93.32	0
4	69.04	71.65	91.16	0
5	62.80	65.79	89.02	0
6	57.07	60.36	86.92	0
7	51.81	55.34	84.84	0
8	47.00	50.68	82.80	0
9	42.59	46.38	80.78	0
10	38.55	42.41	78.80	0

Table Ib: Time Equals 13
Population = 344

0	100.00	100.00	122.04	13
1	91.27	92.11	119.29	13
2	83.24	84.78	116.57	13
3	75.84	77.97	113.89	13
4	69.04	71.65	111.25	13
5	62.80	65.79	108.64	13
6	57.07	60.36	106.07	13
7	51.81	55.34	103.54	13
8	47.00	50.68	101.05	13
9	42.59	46.38	98.59	13
10	38.55	42.41	96.17	13
11	42.59	43.68	97.76	11
12	42.52	42.33	96.92	10
13	42.41	40.99	95.84	9
14	42.26	39.65	94.55	8
15	42.06	38.31	93.07	7
16	41.82	36.98	91.43	6
17	41.53	35.67	89.64	5
18	41.19	34.36	87.73	4
19	40.81	33.07	85.72	3
20	40.39	31.79	83.62	2
21	39.91	30.53	81.45	1
22	39.40	29.28	79.23	0

Table Ic: Time Equals 22
Population = 821

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	448.17	245.96	168.15	7
1	409.05	226.55	164.35	7
2	373.04	208.52	160.61	7
3	339.44	191.77	156.92	7
4	309.44	176.23	153.28	7
5	281.45	161.82	149.69	7
6	255.77	148.47	146.15	7
7	232.21	136.10	142.66	7
8	210.63	124.66	139.22	7
9	190.87	114.08	135.84	7
10	172.79	104.31	132.50	7
11	190.86	107.44	131.25	5
12	190.57	104.12	128.67	4
13	190.09	100.81	125.93	3
14	189.40	97.51	123.06	2
15	188.51	94.23	120.08	1
16	187.41	90.97	117.00	0
17	41.53	35.67	97.93	14
18	41.19	34.36	97.77	13
19	40.81	33.07	97.27	12
20	40.39	31.79	96.47	11
21	39.91	30.53	95.40	10
22	39.40	29.28	94.09	9
23	42.92	29.79	93.54	7
24	42.26	28.50	91.67	6
25	41.55	27.24	89.65	5
26	40.79	26.00	87.52	4
27	40.00	24.79	85.29	3
28	39.16	23.60	82.98	2
29	42.30	23.83	80.65	0

Table Id: Time Equals 33
Population = 1842

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	2008.55	604.96	227.04	3
1	1833.24	557.23	221.92	3
2	1671.83	512.87	216.86	3
3	1523.33	471.68	211.88	3
4	1386.80	433.46	206.96	3
5	1261.39	398.01	202.12	3
6	1146.28	365.17	197.34	3
7	1040.71	334.76	192.63	3
8	943.97	306.62	187.99	3
9	855.40	280.60	183.41	3
10	774.38	256.56	178.25	3
11	855.40	264.26	175.25	1
12	854.09	256.10	170.95	0
13	190.09	100.81	143.24	14
14	189.40	97.51	143.60	13
15	188.51	94.23	142.60	12
16	187.41	90.97	141.59	11
17	186.11	87.73	140.18	10
18	184.61	84.52	138.43	9
19	182.90	81.34	136.37	8
20	180.99	78.19	134.04	7
21	178.88	75.08	131.48	6
22	176.57	72.02	128.72	5
23	192.37	73.26	126.31	3
24	189.40	70.10	123.06	2
25	186.21	67.00	119.71	1
26	182.83	63.96	116.28	0
27	40.00	24.79	97.01	14
28	39.16	23.60	96.54	13
29	42.30	23.83	97.59	11
30	41.29	22.63	96.21	10
31	40.24	21.45	94.59	9
32	39.15	20.31	92.76	8
33	42.03	20.39	91.54	6
34	40.76	19.25	89.22	5
35	39.46	18.14	86.79	4
36	42.14	18.13	84.52	2
37	40.65	17.03	81.83	1
39	39.14	15.97	79.09	0

Table Ie: Time Equals 40
Population = 2646

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	2008.55	604.96	254.08	10
1	1833.24	557.23	248.35	10
2	1671.83	512.87	242.69	10
3	1523.33	471.68	237.11	10
4	1386.80	433.46	231.61	10
5	1261.39	398.01	226.19	10
6	1146.28	365.17	220.84	10
7	1040.71	334.75	215.57	10
8	943.97	306.62	210.37	10
9	855.40	280.60	205.26	10
10	774.38	256.56	200.21	10
11	855.40	264.26	200.55	8
12	854.09	256.10	197.56	7
13	851.90	247.95	194.23	6
14	848.82	239.84	190.60	5
15	844.83	231.77	186.70	4
16	839.92	223.74	182.58	3
17	834.10	215.78	178.27	2
18	827.37	207.88	173.80	1
19	819.72	200.05	169.21	0
20	180.99	78.19	141.50	14
21	178.88	75.08	141.14	13
22	176.57	72.02	140.29	12
23	192.37	73.26	141.35	10
24	189.40	70.10	139.32	9
25	186.21	67.00	136.98	8
26	182.83	63.96	134.38	7
27	179.25	60.97	131.55	6
28	175.49	58.61	128.52	5
29	189.59	58.61	125.85	3
30	185.05	55.65	122.35	2
31	180.34	52.77	118.75	1
32	175.47	49.96	115.09	0
33	42.03	20.39	98.26	13
34	40.76	19.25	97.24	12
35	39.46	18.14	95.91	11
36	42.65	18.13	95.69	9
37	40.65	17.03	93.63	8
38	39.14	15.97	91.41	7
39	41.56	15.88	89.66	5
40	39.86	14.84	87.01	4
41	42.16	14.70	84.53	2
42	40.26	13.69	81.63	1

Table If: Time Equals 50
Population = 5279

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	9001.69	1487.97	343.95	5
1	8216.04	1370.56	336.19	5
2	7492.64	1261.15	328.53	5
3	6827.08	1160.15	320.98	5
4	6215.21	1066.14	313.53	5
5	5653.15	978.96	306.19	5
6	5137.25	898.17	298.95	5
7	4664.12	823.37	291.82	5
8	4230.57	754.16	284.78	5
9	3833.64	690.16	277.85	5
10	3470.54	631.04	271.03	5
11	3833.61	649.97	266.87	3
12	3827.78	629.89	260.92	2
13	3817.97	609.87	254.73	1
14	3804.14	589.91	248.35	0
15	844.33	231.77	207.98	14
16	839.92	223.74	207.76	13
17	834.10	215.73	206.82	12
18	827.37	207.88	205.24	11
19	819.72	200.05	203.08	10
20	811.16	192.31	200.42	9
21	801.69	184.67	197.32	8
22	791.34	177.13	193.83	7
23	862.16	180.19	191.34	5
24	848.81	172.43	186.92	4
25	834.54	164.80	182.29	3
26	819.37	157.31	177.48	2
27	803.33	149.97	172.52	1
28	786.47	142.78	167.46	0
29	189.59	58.61	143.20	13
30	185.05	55.65	141.94	12
31	180.34	52.77	140.23	11
32	175.47	49.96	138.14	10
33	188.37	50.15	137.38	8
34	182.67	47.34	134.35	7
35	176.84	44.62	131.11	6
36	188.86	44.59	128.38	4
37	182.17	41.89	124.60	3
38	175.39	39.28	120.72	2
39	186.26	39.05	116.82	0

Table If: Time Equals 50 (cont.)
Population = 5279

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
40	39.86	14.84	96.93	14
41	42.16	14.70	98.06	12
42	40.26	13.69	96.40	11
43	42.41	13.51	95.84	9
44	40.32	12.53	93.45	8
45	42.28	12.31	91.68	6
46	40.01	11.37	88.81	5
47	41.75	11.12	86.21	3
48	39.32	10.23	83.07	2
49	40.82	9.96	79.93	0

Table IIa: Time Equals 0
Population = 100

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	100.00	100.00	100.00	0
1	90.44	91.35	97.52	0
2	81.71	83.37	95.07	0
3	73.74	76.02	92.67	0
4	66.48	69.25	90.30	0
5	59.87	63.02	87.96	0

Table IIb: Time Equals 7
Population = 1075

0	100.00	100.00	141.91	7
1	90.44	91.35	138.39	7
2	81.71	83.37	134.92	7
3	73.74	76.02	131.50	7
4	66.48	69.25	128.14	7
5	59.87	63.02	124.83	7
6	65.79	67.24	121.57	6
7	59.11	61.07	118.36	5
8	64.80	65.02	115.21	5
9	58.09	58.93	112.10	5
10	63.53	62.61	109.05	4
11	56.82	56.62	106.04	4
12	61.98	60.02	103.09	3
13	55.29	54.15	100.18	3
14	60.16	57.27	97.33	2
15	65.36	60.49	94.53	1
16	58.07	54.38	91.77	1
17	62.92	57.29	89.06	0
18	55.74	51.37	86.40	0

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table IIc: Time Equals 14
Population = 4541

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	604.96	422.07	201.38	5
1	547.12	385.57	196.38	5
2	494.30	351.90	191.46	5
3	446.12	320.87	186.61	5
4	402.20	292.30	181.84	5
5	362.21	266.01	177.14	5
6	397.99	283.81	172.51	4
7	357.62	257.76	167.96	4
8	392.04	274.44	163.48	3
9	351.45	248.73	159.08	3
10	384.35	264.26	154.74	2
11	343.72	238.98	150.48	2
12	374.97	253.32	146.29	1
13	334.47	228.56	142.17	1
14	363.92	241.72	138.12	0
15	65.36	60.49	134.14	8
16	58.07	54.38	130.23	8
17	62.92	57.29	126.39	7
18	55.74	51.37	122.61	7
19	60.22	53.98	118.91	6
20	64.96	56.65	115.27	5
21	57.28	50.59	111.71	5
22	61.59	52.93	108.20	4
23	54.14	47.13	104.77	4
24	58.02	49.17	101.40	3
25	62.08	51.21	98.10	2
26	54.28	45.39	94.86	2
27	57.86	47.12	91.69	1
28	61.57	48.84	88.58	0

Table IId: Time Equals 25
Population = 28,502

Block #	N/ℓ	Density	Unit Price of Housing Services	Age of Buildings
0	3659.82	1781.43	349.03	7
1	3309.88	1627.36	340.37	7
2	2990.34	1485.26	331.84	7
3	2698.84	1354.29	323.44	7
4	2433.17	1233.70	315.17	7
5	2191.27	1122.74	307.03	7
6	2407.67	1197.86	299.01	6
7	2163.46	1087.93	291.12	6
8	2371.68	1158.33	283.36	5
9	2126.14	1049.82	275.72	5
10	2325.21	1115.35	268.21	4
11	2079.39	1008.65	260.82	4
12	2268.42	1069.20	253.56	3
13	2023.43	964.69	246.41	3
14	2201.60	1020.21	239.39	2
15	2392.23	1077.61	232.50	1
16	2125.23	968.73	225.72	1
17	2302.78	1020.66	219.06	0
18	2039.91	915.17	212.52	0
19	364.29	227.85	206.10	8
20	392.97	239.10	199.80	7
21	346.52	213.51	193.61	7
22	372.62	223.41	187.54	6
23	327.51	198.92	181.59	6
24	351.01	207.52	175.75	5
25	375.53	216.16	170.03	4
26	328.36	191.56	164.42	4
27	350.05	198.89	158.92	3
28	372.46	206.16	153.53	2
29	395.55	213.31	148.26	1
30	343.24	187.75	143.09	1
31	363.05	193.56	138.04	0
32	63.34	47.19	133.09	8
33	54.62	41.30	128.25	8
34	57.40	42.33	123.52	7
35	60.18	43.30	118.89	6
36	62.95	44.19	114.37	5
37	65.95	45.01	109.96	4
38	55.97	38.97	105.64	4
39	58.10	39.51	101.44	3
40	60.15	39.95	97.33	2
41	62.11	40.30	93.33	1
42	63.94	40.55	89.42	0

Table IIe: Time Equals 30
Population = 71,802

Block #	N/[illegible]	Density	Unit Price of Housing Service	Age of Buildings
0	22140.65	7518.85	448.17	3
1	20023.61	6868.60	437.05	3
2	18090.53	6268.82	426.10	3
3	16327.05	5716.06	415.31	3
4	14719.83	5207.06	404.69	3
5	13256.43	4738.75	394.23	3
6	14565.60	5055.79	383.94	2
7	13088.17	4591.82	373.81	2
8	14347.84	4888.95	363.84	1
9	12862.40	4430.96	354.03	1
10	14066.70	4707.55	344.39	0
11	12579.61	4257.19	334.90	0
12	2268.42	1069.20	325.57	8
13	2023.43	964.69	316.40	8
14	2201.60	1020.21	307.39	7
15	2392.23	1077.61	298.53	6
16	2125.23	968.73	298.83	6
17	2302.78	1020.66	281.28	5
18	2039.91	915.17	272.88	5
19	2203.85	961.68	264.64	4
20	2377.33	1009.16	256.55	3
21	2096.3[illegible]	901.15	248.60	3
22	2254.23	942.95	240.81	2
23	1981.33	839.57	233.17	2
24	2123.46	875.89	225.67	1
25	2271.85	912.35	218.32	0
26	1986.48	808.53	211.12	0
27	350.05	198.89	204.06	8
28	372.40	206.16	197.14	7
29	395.53	213.31	190.37	6
30	343.24	187.75	183.73	6
31	363.05	193.56	177.24	5
32	383.20	199.18	170.89	4
33	330.43	174.32	164.68	4
34	347.24	178.67	158.60	3
35	364.07	182.75	152.66	2
36	380.61	186.53	146.86	1
37	397.35	189.97	141.19	0
38	338.60	164.49	135.65	0
39	58.10	39.51	130.25	8
40	60.15	39.95	124.97	7
41	62.11	40.30	119.83	6
42	63.94	40.55	114.82	5
43	65.63	40.69	109.93	4
44	54.98	34.70	105.17	4
45	56.08	34.63	100.54	3
46	57.01	34.45	96.03	2
47	57.77	34.17	91.65	1
48	58.32	33.78	87.39	0

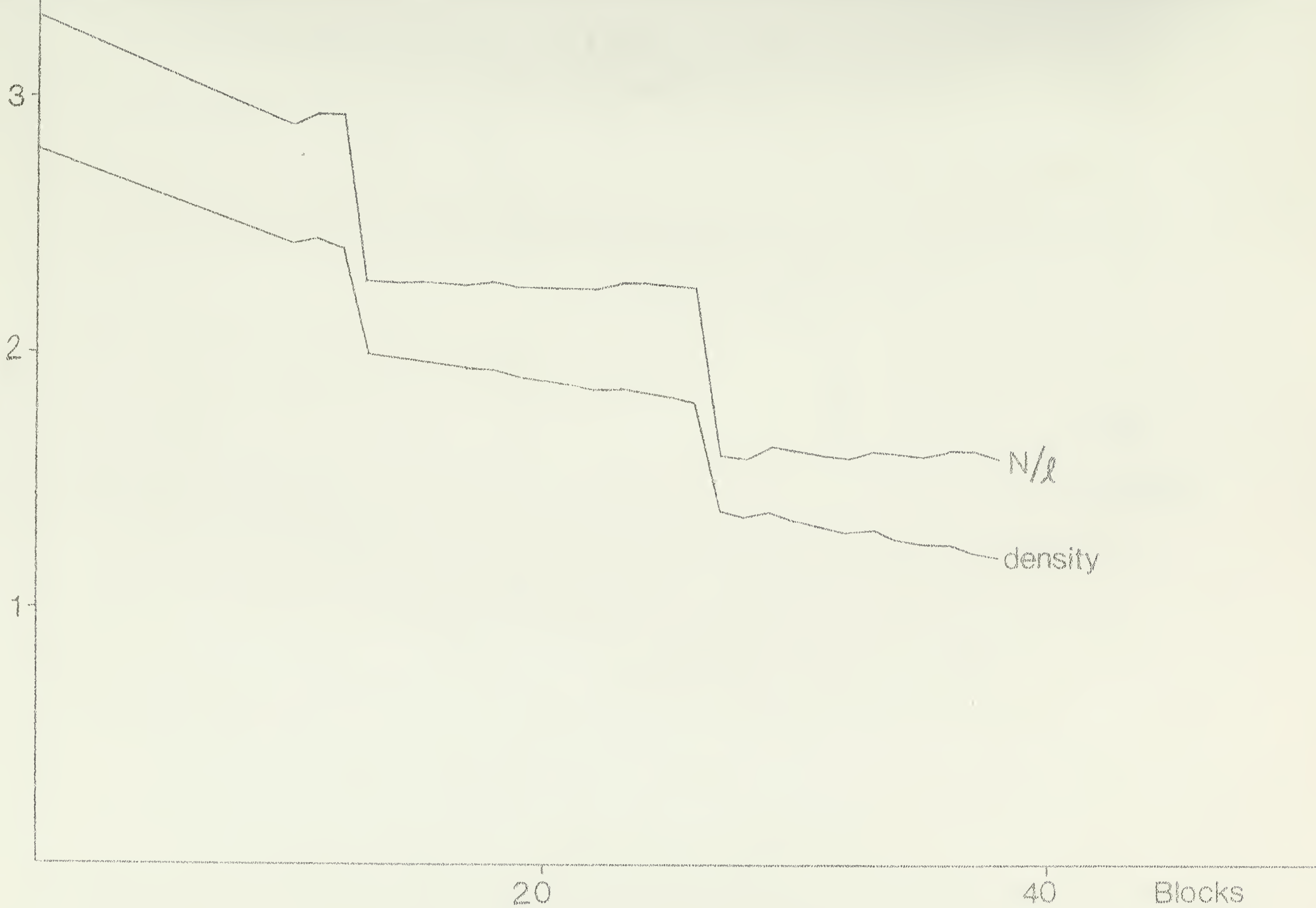

Figure 1.

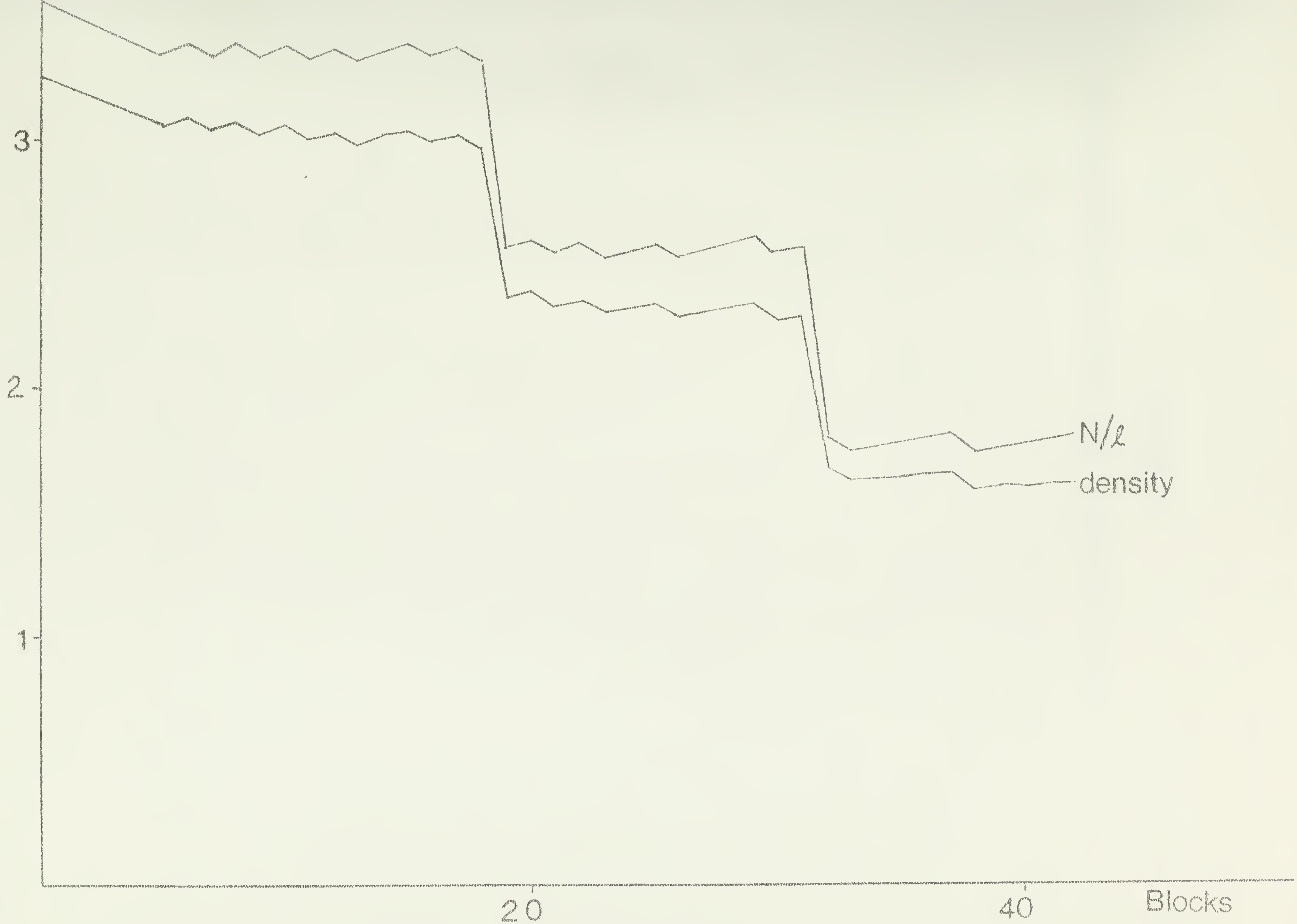

Figure 2.

Footnotes

[1]See Anas (1978), Evans (1975), Fisch (1977), and Muth (1973).

References

A. Anas, Dynamics of urban residential growth, J. Urban Econ., 5, 66-87 (1978).

A. Evans, Rent and housing in the theory of urban growth, J. Regional Sci, 15, 113-25 (1975).

O. Fisch, Dynamics of the housing market, J. Urban Econ., 4, 428-447 (1977).

R. Muth, A vintage model of the housing stock, Pap. Proc. Regional Sci. Assoc., 30, 141-56 (1973).

M/D/90